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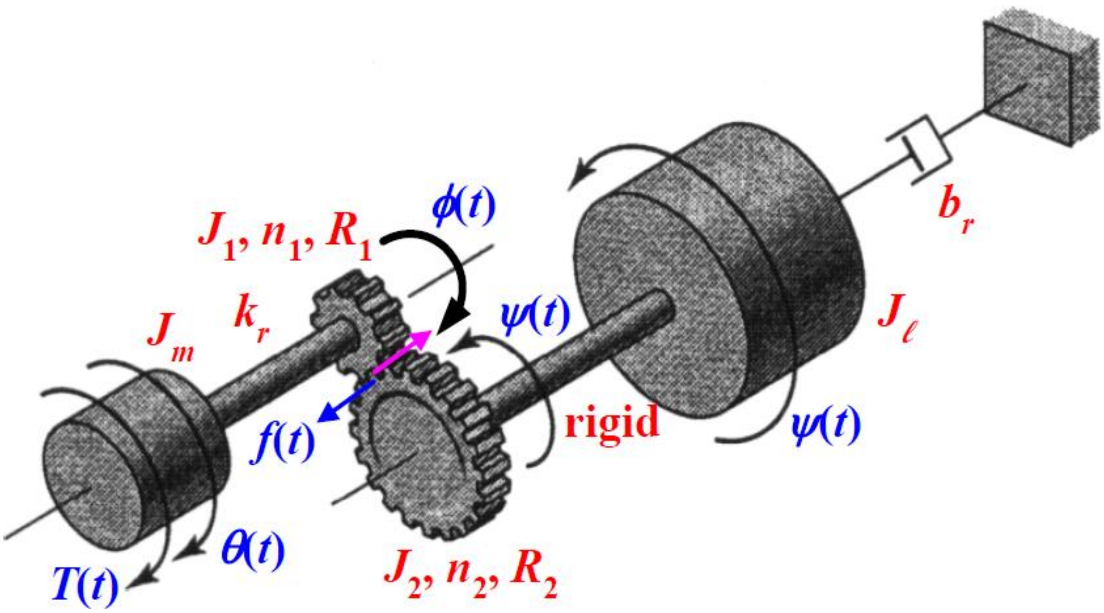
HOMEWORK #4



GEAR- TRAIN MECHANICAL SYSTEM WITH MOTION INPUT

ME 411: System Modeling Analysis and Control

Fall 2010

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1.0 Nomenclature

$K_R = rotational\ stiffness$

$J_m = mass\ moment\ of\ inertia\ of\ motor$

$J_l = mass\ moment\ of\ inertia\ of\ load$

$J_1 = mass\ moment\ of\ inertia\ of\ driven\ spur\ gear 1$

$J_2 = mass\ moment\ of\ inertia\ of\ driven\ spur\ gear 2$

$T(t) = intial\ applied\ torque\ caused\ my\ motion$

$f(t) = meshing\ force\ between\ gears$

$b_r = rotational\ damping\ factor$

$\theta(t) = angular\ displacement\ of\ the\ motor$

$\phi(t) = angular\ displacement\ of\ the\ driven\ gear\ 1$

$\psi(t) = angular\ displacement\ of\ the\ driven\ gear\ 2\ and\ load$

$R_1 = radius\ of\ the\ gear\ 1$

$R_2 = radius\ of\ the\ gear\ 2$

$n_1 = number\ of\ teeth\ in\ gear\ 1$

$n_2 = number\ of\ teeth\ in\ gear\ 2$

$n = gear\ ratio$

$t = time$

$T(s) = transferfucntion\ of\ the\ applied\ torque\ T(t)$

$F(s) = transferfucntion\ of\ the\ meshing\ force\ \ f(t)$

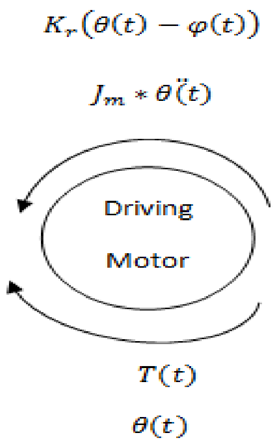
$\Theta(s) = transferfucntion\ of\ the\ angular\ displacement\ of\ the\ motor\ \theta(t)$

$\Phi(s) = transferfucntion\ of\ the\ angular\ displacement\ of\ the\ driven\ gear\ 1\phi(t)$

$\Psi(s) = transferfucntion\ of\ the\ angular\ displacement\ of\ the\ driven\ gear\ 2\ and\ load\ \psi(t)$

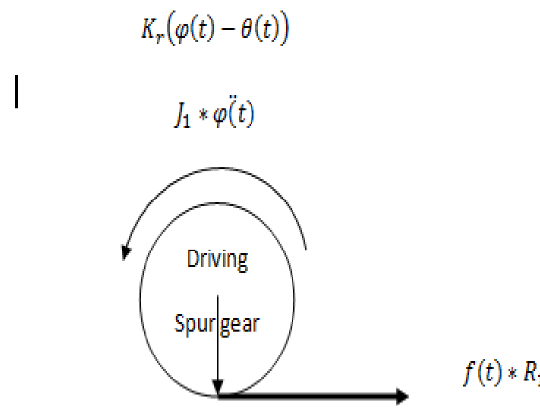
1. FBD

$FBD\ 1.0$



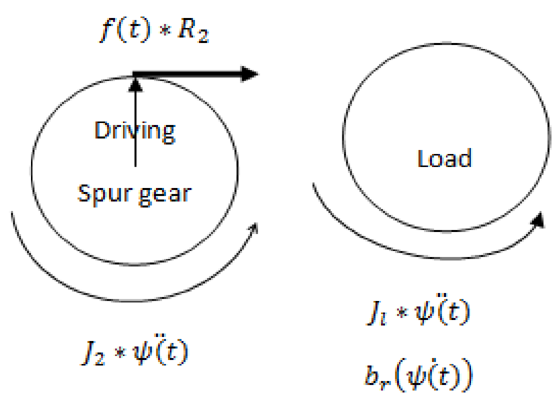
$$J_m * \ddot{\theta}(t) + K_r(\theta(t) - \phi(t)) = T(t)$$

$FBD\ 2.0$



$$J_1 * \ddot{\varphi}(t) + K_r(\phi(t) - \theta(t)) + f(t) * R_1 = 0$$

$FBD\ 3.0$



$$(J_l + J_2) * \ddot{\psi}(t) + b_r(\dot{\psi}(t)) = f(t) * R_2$$

2. **Governing Equations**

With the prescribed motion of the motor $\theta(t)$ as the input, set up the necessary free-body diagrams to derive the equations governing the angular displacements, $\varphi(t)$ and $\psi(t)$, and the contact force, $f(t)$. Find also the applied torque transfer $T(t)$ required to generate the prescribed motion of the motor $\theta(t)$.

- $J_m * \ddot{\theta(t)} + K_r\big(\theta(t) - \phi(t)\big) = T(t)$
- $J_1 * \ddot{\phi(t)} + K_r\big(\phi(t) - \theta(t)\big) + f(t) * R_1 = 0$
- $(J_l + J_2) * \ddot{\psi(t)} + b_r\big(\dot{\psi(t)}\big) - f(t) * R_2 = 0$

3. $\boldsymbol{Laplace\ transfer}$: $\boldsymbol{\mathcal{L}\big(\theta(t)\big) = \Theta(s)}$

- $J_m * \ddot{\theta(t)} + K_r\big(\theta(t) - \phi(t)\big) = T(t)$

  $T(s) = J_m * s^2\Theta(s) + K_r\Theta(s) - K_r\Phi(s)$

- $J_1 * \ddot{\varphi(t)} + K_r\big(\phi(t) - \theta(t)\big) + f(t) * R_1 = 0$

  $J_1 * s^2\Phi(s) + K_r\Phi(s) - K_r\Theta(s) + F(s) * R_1 = 0$

- $(J_l + J_2) * \ddot{\psi(t)} + b_r\big(\dot{\psi(t)}\big) - f(t) * R_2 = 0$

  $J_l s^2\Psi(s) + J_2 s^2\Psi(s) + b_r s\Psi(s) - F(s) * R_2 = 0$

Gear ratio:

$$\frac{\varphi(t)}{\psi(t)} = \frac{R_2}{R_1} = \frac{n_2}{n_1} = n\ and\ R_2 = nR_1\ and\ \Psi(s) = \frac{\Phi(s)}{n}$$

4. Transfer functions

Solving the equations simultaneous:

$$\Phi(s) = \frac{n^2 * K_r * \Theta(s)}{J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r}$$

$$\mathrm{F}(s) = \frac{K_r * \Theta(s) * s * (J_l s + J_2 s + b_r)}{(J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r) * R_1}$$

$$\Psi(s) = \frac{n * K_r * \Theta(s)}{J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r}$$

$$T(s) = J_m * s^2\Theta(s) + K_r\Theta(s) - \frac{n^2 * K_r^2 * \Theta(s)}{J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r}$$

Transfer function:

$$\boldsymbol{G_\phi(s)} = \frac{\boldsymbol{\Phi(s)}}{\boldsymbol{\Theta(s)}} = \frac{n^2 * K_r}{J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r}$$

$$\boldsymbol{G_F(s)} = \frac{\boldsymbol{F(s)}}{\boldsymbol{\Theta(s)}} = \frac{K_r * s * (J_l s + J_2 s + b_r)}{(J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r) * R_1}$$

$$\boldsymbol{G_\psi(s)} = \frac{\boldsymbol{\Psi(s)}}{\boldsymbol{\Theta(s)}} = \frac{n * K_r}{J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r}$$

$$\boldsymbol{G_T(s)} = \frac{\boldsymbol{T(s)}}{\boldsymbol{\Theta(s)}} = J_m * s^2 + K_r - \frac{n^2 * K_r^2}{J_l * s^2 + J_2 * s^2 + b_r * s + n^2 * J_1 * s^2 + n^2 * K_r}$$

5. Steady state response and analysis:

If the input angular displacement is $\theta(t) = \theta_s.1(t) = \begin{Bmatrix} \theta_s & for\ t > 0 \\ 0 & for\ t < 0 \end{Bmatrix}$, where $1(t)$ is the Heaviside unit step function and θ_s is the step constant, determine the resultant steady-state responses of $\phi(t)$, $\psi(t)$, $f(t)$ and $T(t)$, which are defined as: $\phi_{ss} = \mathbf{lim}_{t\to\infty}\,\phi(\boldsymbol{t})$, $\psi_{ss} = \mathbf{lim}_{t\to\infty}\,\psi(\boldsymbol{t})$, $f_{ss} = \mathbf{lim}_{t\to\infty}\,f(\boldsymbol{t})$, $T_{ss} = \mathbf{lim}_{t\to\infty}\,T(\boldsymbol{t})$ respectively. Provide physical interpretation of your results.

Case 1:

$$\rightarrow \phi_{ss} = \lim_{t\to\infty}\phi(t)|\,\theta(t) = \theta_s.1(t)$$

$$\rightarrow \lim_{s\to 0} s\left.\frac{\Phi(s)}{\theta(s)}\right|.\frac{\theta_s}{s}$$

$$\rightarrow \lim_{s\to 0} \cancel{s}\frac{n^2*K_r}{J_l*s^2+J_2*s^2+b_r*s+n^2*J_1*s^2+n^2*K_r}\frac{\theta_s}{\cancel{s}} \rightarrow \boldsymbol{\theta_s}$$

This implies physically that as the time tends to infinity and the body reaches a steady state the output angular motion of the gear1 will be same as that of the input value.

Case 2:

$$\rightarrow \psi_{ss} = \lim_{t\to\infty}\psi(t)|\,\theta(t) = \theta_s.1(t)$$

$$\rightarrow \lim_{s\to 0} s\left.\frac{\Psi(s)}{\theta(s)}\right|.\frac{\theta_s}{s}$$

$$\rightarrow \lim_{s\to 0} \cancel{s}\frac{n*K_r}{J_l*s^2+J_2*s^2+b_r*s+n^2*J_1*s^2+n^2*K_r}\frac{\theta_s}{\cancel{s}} \rightarrow \frac{\boldsymbol{\theta_s}}{\boldsymbol{n}}$$

This implies physically that as the time tends to infinity and the body reaches a steady state the output angular motion of the gear2 will be the ratio of the input value over the gear ration. This means the gear teeth, radius playa very important role in power transmission.

Case 3:

$$\rightarrow f_{ss} = \lim_{t\rightarrow\infty} f(t) | \, \theta(t) = \theta_s . 1(t)$$

$$\rightarrow \lim_{s\rightarrow 0} s \left.\frac{F(s)}{\Theta(s)}\right| . \frac{\theta_s}{s}$$

$$\rightarrow \lim_{s\rightarrow 0} \cancel{s} \frac{K_r*(J_1 s+J_2 s+b_r)*s}{(J_l*s^2+J_2*s^2+b_r*s+n^2*J_1*s^2+n^2*K_r)*R_1} \frac{\theta_s}{\cancel{s}} = \mathbf{0}$$

This implies physically that as the time tends to infinity and the body reaches a steady state, the meshing force between the teeth of the gear becomes zero an ideal stage.

Case 4:

$$\rightarrow T_{ss} = \mathbf{lim}_{t\rightarrow\infty} T(\boldsymbol{t}) | \, \theta(t) = \theta_s . 1(t)$$

$$\rightarrow \mathbf{lim}_{s\rightarrow 0} s \left.\frac{T(\boldsymbol{s})}{\theta(s)}\right| . \frac{\theta_s}{s}$$

$$\rightarrow \mathbf{lim}_{s\rightarrow 0} \cancel{s} \left(J_m * s^2 + K_r - \frac{n^2*K_r^2}{J_l*s^2+J_2*s^2+b_r*s+n^2*J_1*s^2+n^2*K_r} \right) \frac{\theta_s}{\cancel{s}} \rightarrow \boldsymbol{\theta_s * (K_r - 1)}$$

6. **MATLAB solutions for the zeros and poles (i.e., the characteristic roots) for four transform functions. Assume the following data:**

J_m	J_l	J_1	J_2	R_1	R_2	K_r	b_r
[kg-m^2]	[kg-m^2]	[kg-m^2]	[kg-m^2]	m	m	[N-m]	[N-m]/s
0.04	0.08	0.01	0.02	0.1	0.2	0.3	0.2

Definition 1: *zeros*

1. The value(s) for *z* where the *numerator* of the transfer function equals zero
2. The complex frequencies that make the overall gain of the filter transfer function zero.

Definition 2: *poles*

1. The value(s) for *z* where the *denominator* of the transfer function equals zero
2. The complex frequencies that make the overall gain of the filter transfer function infinite.

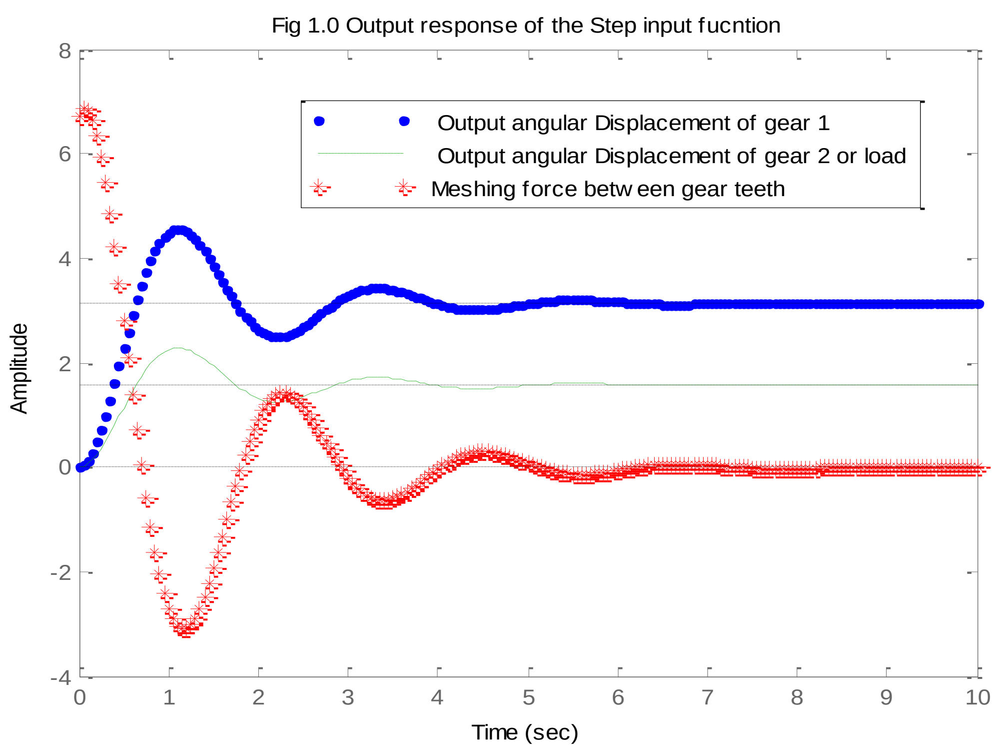
Transfer Function	***Zeros*** (z)	***Poles*** (p)
$G_{\phi}(s) = \dfrac{1.20}{0.140 * s^2 + 0.2 * s + 1.20}$		- 0.7143 + 2.8392i - 0.7143 - 2.8392i
$G_{\mathrm{F}}(s) = \dfrac{0.3 * s^2 + 0.6 * s}{0.140 * s^2 + 0.2 * s + 1.20}$	0 -2	- 0.7143 + 2.8392i - 0.7143 - 2.8392i
$G_{\Psi}(s) = \dfrac{0.6}{0.140 * s^2 + 0.2 * s + 1.20}$		- 0.7143 + 2.8392i - 0.7143 - 2.8392i
$G_T(s) = \dfrac{0.0056 * s^4 + 0.008 * s^3 + 0.09 * s^2 + 0.06 * s}{0.140 * s^2 + 0.2 * s + 1.20}$	0 -0.37 + 3.9275i -0.37 - 3.9275i -0.6885	- 0.7143 + 2.8392i - 0.7143 - 2.8392i

7. <u>MATLAB solutions for transient responses</u>. Use MATLAB to find the responses $\phi(t)$, $\psi\ (t)$ and $f\ (t)$ due to various prescribed motions $\theta\ (t)$ listed below. For each case, plot the time histories of $\phi(t)$, $\psi\ (t)$, $f\ (t)$ and $\theta\ (t)$ on the same graph. For the first three sub cases, determine also the corresponding poles and residues using the MATLAB command: **residue**. Data of additional parameters needed for calculations are tabulated below.

θ_s	θ_r	θ_δ	θ_p	t_p	θ_{tr}	t_{tr}
[rad]	[rad/s]	[deg-s]	[rad]	[s]	[rad]	[s]
π	0.1π	π	π	5	π	5

<u>7.1 Step input</u>: $\theta(t) = \theta_s.1(t) = \begin{Bmatrix} \theta_s & for\ t > 0 \\ 0 & for\ t < 0 \end{Bmatrix}$, *where θ_s is the step constant.*

Compute and plot the results for a duration of 10 s. Determine also the poles and residues of $\Phi(s)$, $\psi(s)$ and $F(s)$, respectively, corresponding to this input motion. Use the MATLAB command: **ltiview** *to determine the peak amplitudes, maximum overshoots in %, peak times, 0-100% rise times, ±2% settling times and steady-state values of the responses $\varphi\ (t)$, $\psi\ (t)$ and $f\ (t)$, respectively. Compare the graphical results with those of Part (d).*



$$\Phi(s) = \frac{188.4955592}{7 * s^2 + 10 * s + 60}$$

RESIDUE: 0 - 4.7414i
0 + 4.7414i
POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i

$$\Psi(s) = \frac{94.24777962}{7 * s^2 + 10 * s + 60}$$

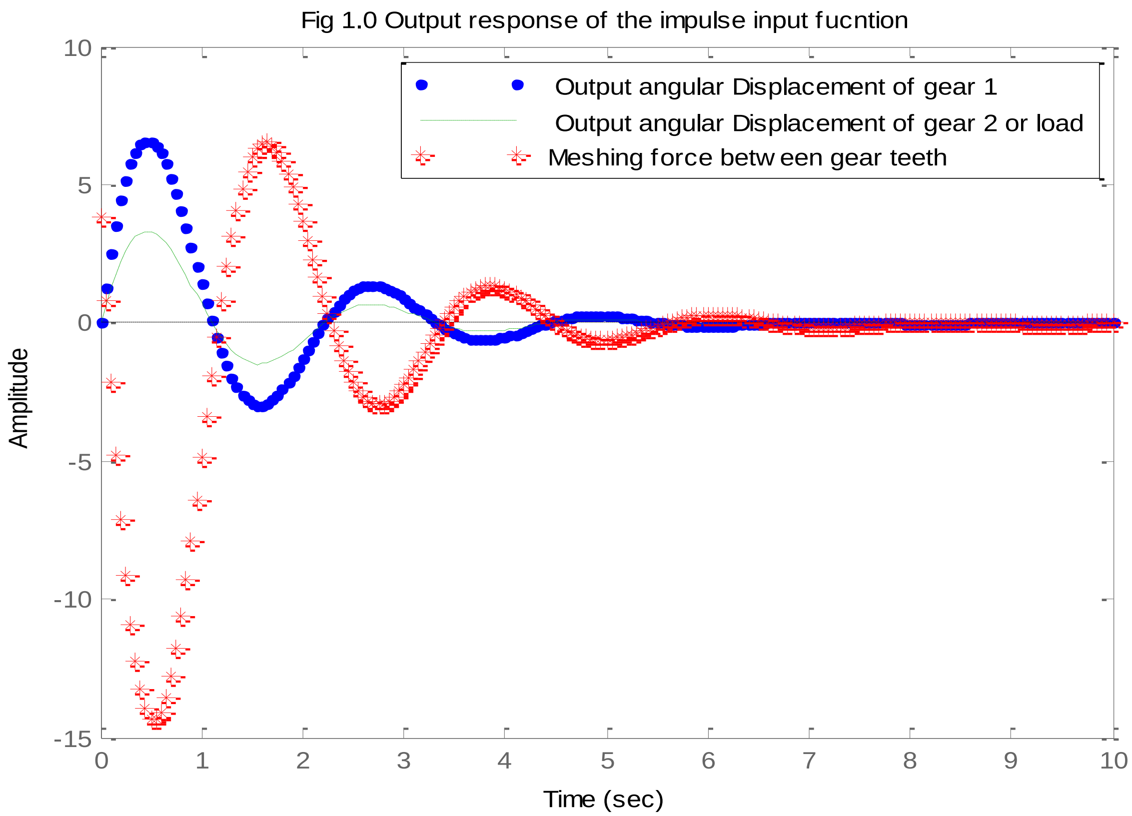
RESIDUE: 0 - 2.3709i
0 + 2.3709i
POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i

$$F(s) = \frac{47.12388981 s^2 + 94.24777962 s}{7 * s^2 + 10 * s + 60}$$

RESIDUE: 1.9233 +10.6447i
1.9233 -10.6447i
POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i

Output	***Peak Amplitudes***	***Maximum Overshoots in %***	***Peak Times***	***0-100% Rise Times***	***±2% Settling Times***	***Steady State***	***Steady State (d)***
$\phi(t)$	*4.57*	*45.4*	*1.1*	*0.6408*	*4.8335*	*3.14*	*pi*
$\psi(t)$	*2.28*	*45.4*	*1.1*	*0.6408*	*4.8335*	*1.57*	*pi/2*
$f(t)$	*6.85*	*infinity*	*0.05*	*0.7039*	*4.8965*	*0*	*0*

7.2 Impulse input: $\theta(t) = \theta_\delta . \delta(t) = \begin{Bmatrix} \infty & for\ t = 0 \\ 0 & for\ t \neq 0 \end{Bmatrix}$, *where* $\delta(t)$ *is the unit impulse function (also called Dirac Delta function) and constant* θ_δ *is the impulse strength. Compute and plot the results for a duration of 10 s. Determine also the poles and residues of Φ(s), ψ(s) and F(s), respectively, corresponding to this input motion. Use the MATLAB command:* **ltiview** *to determine the peak amplitudes, peak times and ±2% settling times of the responses φ (t), ψ (t) and f (t), respectively.*



$$\Phi(s) = \frac{188.4955592}{7 * s^2 + 10 * s + 60}$$

RESIDUE: 0 - 4.7414i

0 + 4.7414i

POLES: -0.7143 + 2.8392i

-0.7143 - 2.8392i

$$\Psi(s) = \frac{94.24777962}{7 * s^2 + 10 * s + 60}$$

RESIDUE: 0 - 2.3709i
0 + 2.3709i

POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i

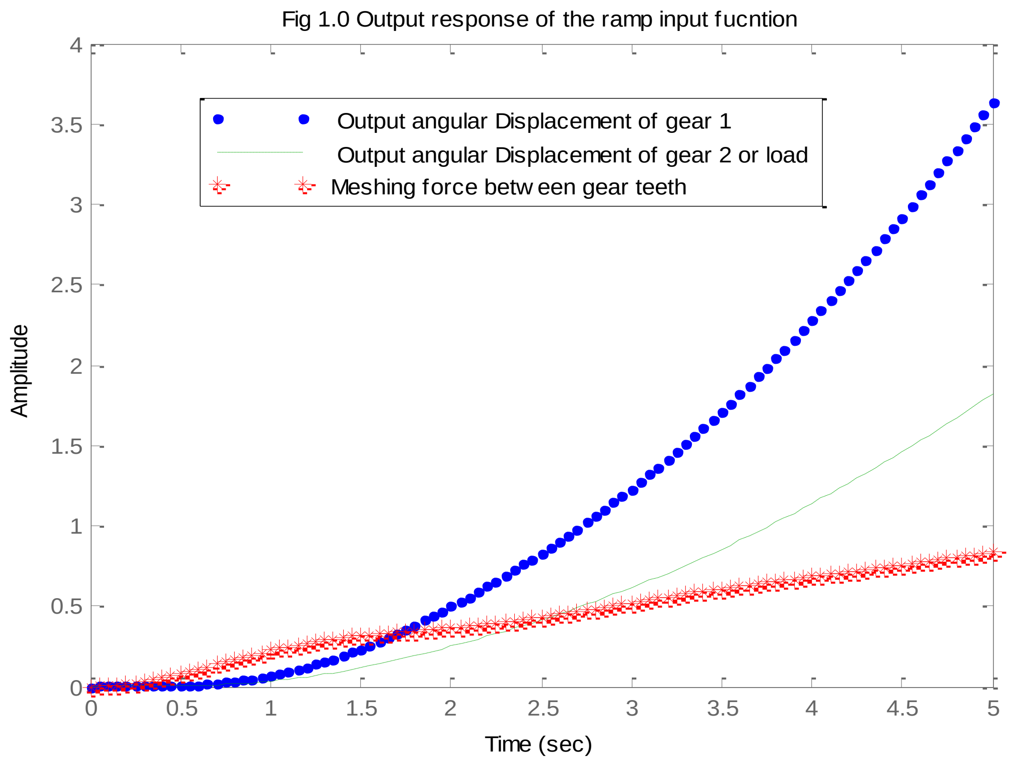
$$F(s) = \frac{47.12388981 s^2 + 94.24777962 s}{7 * s^2 + 10 * s + 60}$$

RESIDUE: 1.9233 +10.6447i
1.9233 -10.6447i

POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i

Output	***Peak Amplitudes***	***Peak Times***	***±2% Settling Times***	***Steady State***	***Steady State (d)***
$\phi(t)$	*6.58*	*0.45*	*5.3*	*0*	*pi*
$\psi(t)$	*3.29*	*0.45*	*5.3*	*0*	*pi/2*
$f(t)$	*-14.3*	*0.55*	*5.36*	*0*	*0*

7.3 Ramp input: $\theta(t) = \theta_r t.1(t) = \begin{Bmatrix} \theta_r t & for\ t > 0 \\ 0 & for\ t < 0 \end{Bmatrix}$, *where the constant* θ_r *is the ramp slope. Compute and plot the results for a duration of 5 s. Determine also the poles and residues of Φ (s), ψ (s) and F (s), respectively, corresponding to this input motion.*



$$\Phi(s) = \frac{18.84955592}{s^3 * (7.* s^2 + 10.* s + 60.)}$$

RESIDUE: 0.0262 + 0.0487i
0.0262 - 0.0487i
-0.0523
0.3140

POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i
0
0

$$\Psi(s) = \frac{9.424777962}{s^3 * (7. * s^2 + 10. * s + 60.)}$$

RESIDUE: 0.0131 + 0.0244i
0.0131 - 0.0244i
-0.0262
0.1571

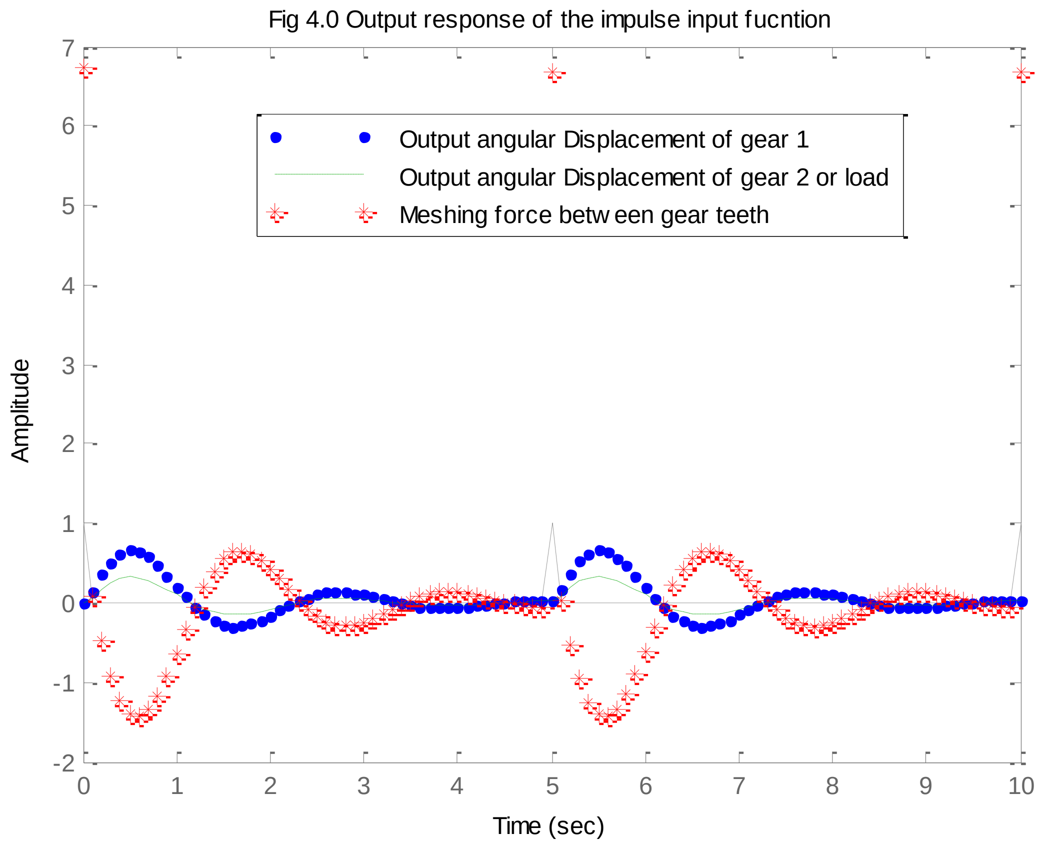
POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i
0
0

$$F(s) = \frac{4.712388981 * (s + 2.)}{s * (7. * s^2 + 10. * s + 60.)}$$

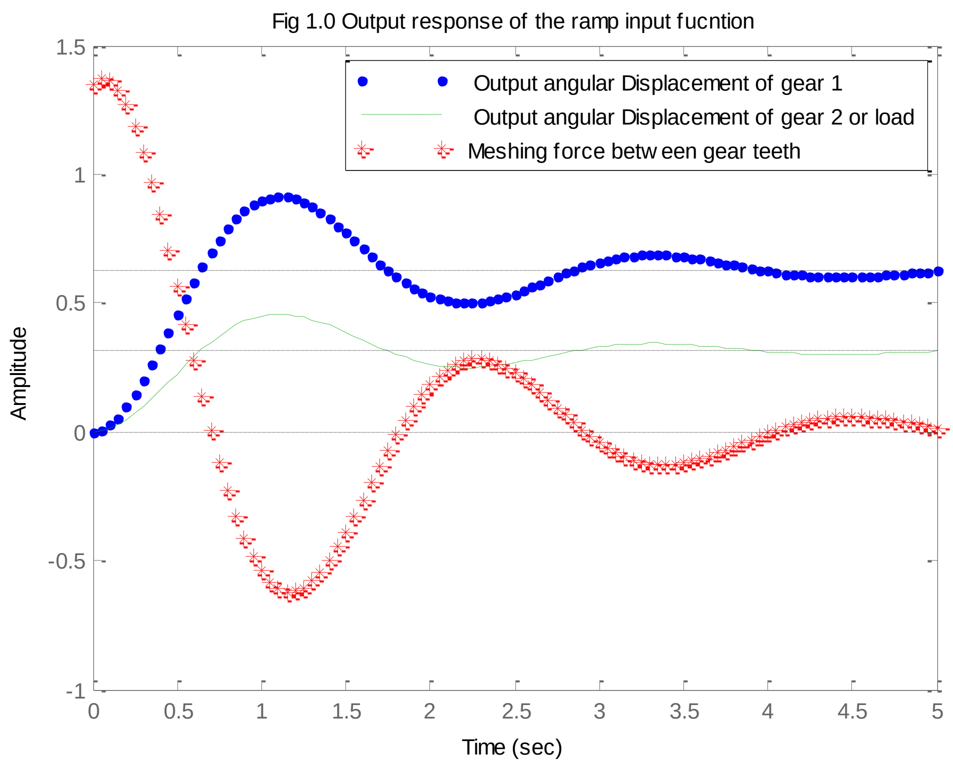
RESIDUE: -0.0785 - 0.0988i
-0.0785 + 0.0988i
0.1571

POLES: -0.7143 + 2.8392i
-0.7143 - 2.8392i
0

7.4 Pulse input: $\theta(t) = \begin{cases} 0 & t_p < t \\ \theta_p \text{ when } 0 < t < t_p \\ 0 & t < 0 \end{cases}$, *where the two constant* θ_p *and* t_p *are the pulse strength and pulse duration, respectively. Compute and plot the results for duration of 10s.*



7.5 Terminated –ramp input: $\theta(t) = \begin{Bmatrix} \theta_{tr} & t_{tr} < t \\ \frac{\theta_{tr}}{t_{tr}} t \text{ when } & 0 < t < t_{tr} \\ 0 & t < 0 \end{Bmatrix}$, *,where the two constant* θ_{tr} *and* t_{tr} *are the terminated step and ramp duration, respectively. Compute and plot the results for duration of 5s.*



8. MATLAB solutions for applied torque. Use MATLAB to find the applied torque $T(t)$ needed to generated the prescribed motions given in Part (e). If you encounter difficulty, state the reason why.

$$T(s) = \frac{.87964 * s^4 + 1.2566 * s^3 + 14.137 * s^2 + 9.424 * s}{7s^2 + 10s + 60}$$

$$T(t) = \frac{1616331 * dirac(t)}{1715000} - \frac{dirac(t,1)}{245000} + \frac{21991 * dirac(t,2)}{175000} - \frac{73 * \left(cos\left(\frac{395^{\frac{1}{2}} * t}{7}\right) + \frac{67885537 * 395^{\frac{1}{2}} * sin\left(\frac{395^{\frac{1}{2}} * t}{7}\right)}{28835} \right)}{1200500 * exp\left(\frac{5*t}{7}\right)}$$

Unable to evaluate the function, Numerical Solutions out of range.

9. Appendix

MATLAB Code for Unit Step input

```
clear all
clc
t=0:0.05:10;

Phi_num=188.495592;
Phi_den=[7 10 60];
transfer_Phi = tf(Phi_num, Phi_den);

Psi_num=94.24777962;
Psi_den=[7 10 60];
transfer_Psi = tf(Psi_num, Psi_den);


force_num=[47.12388981 94.2477962 0];
force_den=[7 10 60];
transfer_Force = tf(force_num, force_den);

step(transfer_Phi,'b.',transfer_Psi,'g--',transfer_Force,'r*',t)
title('Fig 1.0 Output response of the Step input fucntion')
legend('    Output angular Displacement of gear 1','    Output angular
Displacement of gear 2 or load','   Meshing force between gear teeth')

 %Rise time (0-100%)
 %settling time (+-2%)
         %SettlingTime — Settling time
         %SettlingMin — Minimum value of y once the response has risen
         %SettlingMax — Maximum value of y once the response has risen
         %Overshoot — Percentage overshoot (relative to yfinal)
         %Undershoot — Percentage undershoot
         %Peak — Peak absolute value of y
         %PeakTime — Time at which this peak is reached

S_Phi = stepinfo(transfer_Phi,'RiseTimeLimits',[0,1])
S_Psi = stepinfo(transfer_Psi,'RiseTimeLimits',[0,1])
S_Force = stepinfo(transfer_Force,'RiseTimeLimits',[0,1])
```

MATLAB Code for Impulse input

```
clear all
clc
t=0:0.05:10;

Phi_num=188.495592;
Phi_den=[7 10 60];
transfer_Phi = tf(Phi_num, Phi_den);

Psi_num=94.24777962;
Psi_den=[7 10 60];
transfer_Psi = tf(Psi_num, Psi_den);
```

```
    force_num=[47.12388981 94.2477962 0];
    force_den=[7 10 60];
    transfer_Force = tf(force_num, force_den);

    impulse(transfer_Phi,'b.',transfer_Psi,'g--',transfer_Force,'r*',t)
    title('Fig 1.0 Output response of the impulse input fucntion')
    legend('    Output angular Displacement of gear 1','    Output angular
Displacement of gear 2 or load','   Meshing force between gear teeth')
```

MATLAB Code for Ramp input

```
    clear all
    clc
    t=0:0.05:5;

    Phi_num=18.8495592;
    Phi_den=[7 10 60 0 0];
    transfer_Phi = tf(Phi_num, [Phi_den 0]);

    Psi_num=9.424777962;
    Psi_den=[7 10 60 0 0];
    transfer_Psi = tf(Psi_num, [Psi_den 0]);


    force_num=[4.712388981 9.42477962];
    force_den=[7 10 60 0];
    transfer_Force = tf(force_num, [force_den 0]);

    impulse(transfer_Phi,'b.',transfer_Psi,'g--',transfer_Force,'r*',t)
    title('Fig 1.0 Output response of the ramp input fucntion')
    legend('    Output angular Displacement of gear 1','    Output angular
Displacement of gear 2 or load','   Meshing force between gear teeth')
```

MATLAB Code for Pulse input

```
clear all
clc
[theta,t] = gensig('Pulse',5,10,0.1);

    Phi_num=188.495592;
    Phi_den=[7 10 60];
    transfer_Phi = tf(Phi_num, Phi_den);

    Psi_num=94.24777962;
    Psi_den=[7 10 60];
    transfer_Psi = tf(Psi_num, Psi_den);

    force_num=[47.12388981 94.2477962 0];
    force_den=[7 10 60];
    transfer_Force = tf(force_num, force_den);
    lsim(transfer_Phi,'b.',transfer_Psi,'g--',transfer_Force,'r*',theta,t)
    title('Fig 4.0 Output response of the impulse input fucntion')
    legend('    Output angular Displacement of gear 1','    Output angular
Displacement of gear 2 or load','    Meshing force between gear teeth')
```